Curves of Constant Precession

Paul D. Scofield

1. INTRODUCTION. Given initial position and direction, the flight-path of a ship in Euclidean space is completely determined by how much it turns and how much it twists at each odometer reading. This is an intuitive interpretation of the Fundamental Theorem for Space Curves, which states that curvature κ and torsion τ, as functions of arclength s, determine a space curve uniquely up to rigid motion. This statement of the Fundamental Theorem ([**14**], §1–8) should be tempered with the reservations expressed by Nomizu [**12**] and Wong & Lai [**15**].

Given a parametric space curve, there are well-known formulae for the arclength, curvature, and torsion (as functions of the parameter). Given two functions of one parameter (potentially curvature and torsion parametrized by arc-length) one might like to find a parametrized space curve for which the two functions are the curvature and torsion. This activity, called "solving natural equations" ([**14**], §1–10), is generally achieved by solving Riccati equations like $dw/ds = -i\tau/2 - i\kappa w + i\tau w^2/2$.

Although the solution generally exists, it usually cannot be obtained explicitly. Euler [**6**] found explicit integral formulae for plane curves (where $\tau \equiv 0$) through direct geometric analysis. Hoppe [**9**] developed a general method for solving the natural equations for space curves by solving Riccati equations through a complicated sequence of integral transformations. He digressed to obtain formulae for the tangent, normal, and binormal indicatrices for general helices and essentially for curves of constant precession. Enneper [**5**] obtained explicit closed-form solutions for helices on revolved conic sections through direct geometric analysis.

A curve of constant precession is defined by the property that as the curve is traversed with unit speed, its centrode revolves about a fixed axis with constant angle and constant speed. In this paper we obtain an arclength-parametrized closed-form solution of the natural equations for curves of constant precession through direct geometric analysis. As part of this analysis, we obtain a new theorem for curves of constant precession analogous with Lancret's Theorem for general helices. We provide the first rendering of a curve of constant precession. We also note for the first time that curves of constant precession lie on circular hyperboloids of one sheet and have closure conditions that are simply related to their arclength, curvature, and torsion. These are 3-type curves, except one family of closed 2-type curves (when $\omega = \sqrt{3}\mu$; see [**2**], [**3**], and [**1**]).

Given a closed C^3 curve in space, it is rather obvious that the curvature and torsion functions will be periodic functions of the arclength, with period equal the total arclength. This is a necessary condition but, as the circular helices (κ and τ both constant) show, not a sufficient condition that integral curves be closed. Efimov [**4**] and Fenchel [**7**] independently formulated

The Closed Curve Problem. *Find (explicit) necessary and sufficient conditions that determine when, given two periodic functions $\kappa(s)$ and $\tau(s)$ with the same period L, the integral curve is closed.*

This natural problem in elementary differential geometry remains open, despite implicit solutions by Schmeidler [**13**] and Hwang [**10**]. Fenchel warned that there may be no simple solution. Our investigation of curves of constant precession began in an effort to find closure conditions for some collection of pairs of simple periodic functions like $\kappa(s) = \omega \cos \mu s$ and $\tau(s) = \omega \sin \mu s$.

2. PLANE CURVES. Here we set out Euler's well-known integral solutions of the natural equations for plane curves ([**14**], p. 26). We will designate coordinates and geometric invariants of plane curves by subscript π. Identifying the angle between the tangent line to the curve and the x-axis as

$$\varphi_\pi = \int \kappa_\pi \, ds_\pi,$$

it follows that

$$x_\pi = \int \cos \varphi_\pi \, ds_\pi \quad \text{and} \quad y_\pi = \int \sin \varphi_\pi \, ds_\pi$$

solve natural equations of the form

$$\kappa_\pi = \kappa_\pi(s_\pi) \quad \text{and} \quad \tau_\pi \equiv 0.$$

If we change a constant of integration, we rotate or translate the curve.

Still, it is a rare curve for which both κ is a simple function and the above integrals can be evaluated in closed form with elementary functions. Among the simplest are the circle, the logarithmic spiral, the circle involute, and the epicycloid ([**14**], pp. 26–28). Enneper [**5**] showed that each of these is the projection along the axis of symmetry of a curve of constant slope (helix) on a conic surface of revolution: a circular cylinder, a cone, a paraboloid, and a sphere.

3. CURVES OF CONSTANT SLOPE (HELICES). Here we set out the integral solution of the natural equations for curves of constant slope or general helices ([**14**], pp. 33–35), and we set out an explicit parametrization for spherical helices, never appealing to the solution of a Riccati equation. A *curve of constant slope* or *helix* is defined by the property that the tangent makes a constant angle θ with a fixed line l. We have its natural equations by

The Theorem of Lancret [11]. *A necessary and sufficient condition that a curve be of constant slope is that the ratio of curvature to torsion be constant.*

In proving the theorem, it is observed that the constant slope and the constant ratio are related by

$$\kappa / \tau = \tan \theta, \text{ constant}.$$

Taking l as the z-axis, it is easy to observe that $dz = \cos \theta \, ds$. Moreover, the projection of the curve onto the xy-plane has arclength element $ds_\pi = \sin \theta \, ds$ and curvature $\kappa_\pi = \kappa \csc^2 \theta$ (relating the radii of a helical osculating circle and the planar osculating circle of its projection). Then using Euler's planar solution,

$$\varphi_\pi = \int \kappa_\pi \, ds_\pi = \csc \theta \int \kappa \, ds,$$

so

$$x(s) = \sin\theta \int_0^s \cos\left[\csc\theta \int_0^{s_1} \kappa(s_2)\,ds_2\right] ds_1$$

$$y(s) = \sin\theta \int_0^s \sin\left[\csc\theta \int_0^{s_1} \kappa(s_2)\,ds_2\right] ds_1$$

$$z(s) = s\cos\theta.$$

General helices are precisely the geodesics on general cylinders generated by lines parallel with l. A general cylinder is the rectifying developable of its helices.

We will want a parametrization for spherical helices because the tangent indicatrix of a curve of constant precession will prove to be a spherical helix. In anticipation, we will designate the coordinates and arclength of spherical helices by subscript t. Struik ([**14**], pp. 34–35) shows that for a helix on a sphere of radius r making an angle θ with the z-axis, the projection onto the xy-plane is an epicycloid with fixed radius $a = r\cos\theta$ and rolling radius $b = r\sin^2(\theta/2)$. Substituting these into his epicycloid parametrization (p. 27), we obtain

$$x_t(\psi) = \frac{r}{2}(1+\cos\theta)\cos\psi - \frac{r}{2}(1-\cos\theta)\cos\frac{1+\cos\theta}{1-\cos\theta}\psi$$

$$y_t(\psi) = \frac{r}{2}(1+\cos\theta)\sin\psi - \frac{r}{2}(1-\cos\theta)\sin\frac{1+\cos\theta}{1-\cos\theta}\psi$$

$$z_t(\psi) = r\sin\theta\cos\frac{\cos\theta}{1-\cos\theta}\psi,$$

where

$$s_\pi = r\sin\theta\tan\theta\cos\frac{\cos\theta}{1-\cos\theta}\psi,$$

$$s_t = r\tan\theta\cos\frac{\cos\theta}{1-\cos\theta}\psi.$$

The spherical helix has an arc of length $2r\tan\theta$ between heights $z = \pm r\sin\theta$ beyond which no tangent to the sphere makes an angle as small as θ with the z-axis. The parametric extension gives a sequence of arcs which join in cusps at their endpoints. This piecewise smooth curve is closed if and only if $\cos\theta$ is rational. All arcs of a spherical helix with $\cos\theta = 8/17$ are rendered in Figure 1.

4. CURVES OF CONSTANT PRECESSION. Here we characterize curves of constant precession. We will denote the moving orthonormal frame of tangent, normal, and binormal vectors by $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$, and we will differentiate with respect to arclength, using the *Frenet equations*

$$\begin{aligned}\mathbf{t}' &= \kappa\mathbf{n}\\ \mathbf{n}' &= -\kappa\mathbf{t} + \tau\mathbf{b}\\ \mathbf{b}' &= -\tau\mathbf{n}\end{aligned}$$

([**8**], [**14**], §1-6). Let $\mathbf{C} = \tau\mathbf{t} + \kappa\mathbf{b}$ denote the *centrode*, the Frenet frame's axis of instantaneous rotation ([**14**], §1-6, Exercise 18, and [**7**]). Fix arbitrary constants $\omega > 0$, μ, and $\alpha = \sqrt{\omega^2 + \mu^2}$. Set $\mathbf{A} = \mathbf{C} \pm \mu\mathbf{n}$ and fix the line l parallel with $\mathbf{A}(0)$. We use $\angle(\bullet,\bullet)$ to denote the angle between two vectors.

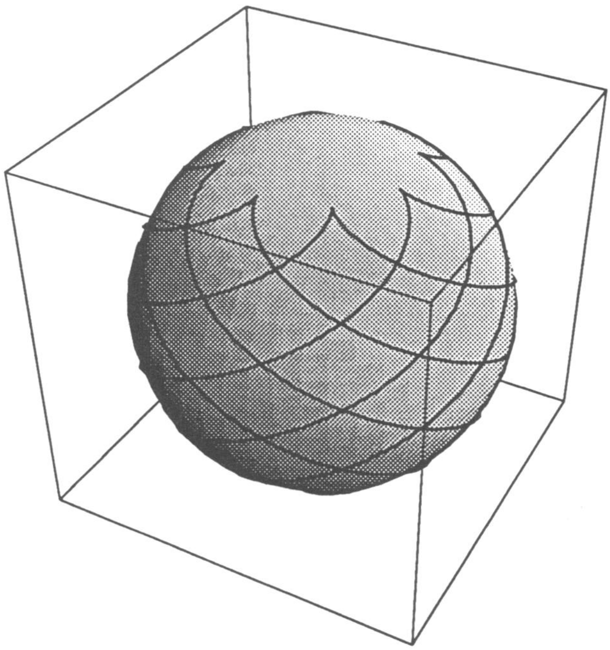

Figure 1. Sixteen arcs of a helix on the unit sphere with $\cos\theta = 8/17$. They form the tangent indicatrix of the curve in Figure 2. (Visualization assisted by The Geometry Center at the University of Minnesota.)

Lemma. *The following are equivalent*:

$$\text{(i) } |\mathbf{C}| = \omega$$

$$\text{(ii) } \angle(\mathbf{C}, \mathbf{A}) = \cos^{-1}\frac{\omega}{\alpha}$$

$$\text{(iii) } |\mathbf{n}'| = \omega$$

$$\text{(iv) } \angle(\mathbf{n}, \mathbf{A}) = \cos^{-1}\frac{\mu}{\alpha}$$

$$\text{(v) } |\mathbf{A}| = \alpha.$$

Proof: Since $|\mathbf{C}|^2 = \kappa^2 + \tau^2 = |\mathbf{n}'|^2$ and $|\mathbf{A}|^2 = \kappa^2 + \tau^2 + \mu^2$, it is clear that (i), (iii) and (v) are equivalent. Interpreting (ii) as

$$\kappa^2 + \tau^2 = \mathbf{C} \cdot \mathbf{A} = \frac{\omega}{\alpha}\sqrt{\kappa^2 + \tau^2}\sqrt{\kappa^2 + \tau^2 + \mu^2}$$

implies that (i) is equivalent to (ii), and interpreting (iv) as

$$\mu = \mathbf{n} \cdot \mathbf{A} = \frac{\mu}{\alpha}|\mathbf{n}|\,|\mathbf{A}|$$

implies that (iv) is equivalent to (v). Q.E.D.

Lemma. *Given any of* (i)–(v), *the following are equivalent*:

$$\text{(vi) } |\mathbf{C}'| = |\omega\mu|$$

(vii) $\mathbf{A}$ *is parallel with* l.

Proof: Since $\mathbf{A}' = \mathbf{C}' \pm \mu\mathbf{n}'$,

$$\mathbf{A}' = 0 \Leftrightarrow \mathbf{C}' = \mp\mu\mathbf{n}' \Leftrightarrow |\mathbf{C}'| = |\mu|\,|\mathbf{n}'|.$$

Thus, it follows from (iii) and (v) that (vi) and (vii) are equivalent. Q.E.D.

A *curve of constant precession* is defined (somewhat redundantly) by the property that, as it is traversed with unit speed, its centrode revolves about a fixed line l in space (the *axis*) with constant angle and constant speed. As a consequence, its Frenet frame precesses about l, while its principal normal revolves about l with constant complementary angle and constant speed. We have its natural equations by the following analogy with Lancret's Theorem.

Theorem 1. *A necessary and sufficient condition that a curve be of constant precession is that $\kappa(s) = \omega \sin \mu s$ and $\tau(s) = \omega \cos \mu s$, up to reflection or phase shift of arclength, for constants ω and μ.*

Proof: Conditions (v) and (vii) are true if and only if $\mathbf{A}' = 0$, but

$$\mathbf{A}' = (\tau' - \mu\kappa)\mathbf{t} + (\kappa' + \mu\tau)\mathbf{b}$$

and uniqueness of solutions of pairs of linear equations imply that $\mathbf{A}' = 0$ if and only if $\kappa(s) = \omega \sin \mu s$ and $\tau(s) = \omega \cos \mu s$ (up to reflection or phase shift). Q.E.D.

5. SOLVING THE NATURAL EQUATIONS. Here, without solving a Riccati equation but using results from Sections 3 and 4, we obtain an arclength parametrization for curves of constant precession. Condition (iv) of the lemmata in Section 4 implies, since $\mathbf{t}' = \kappa\mathbf{n}$, that $\mathbf{t}$ is a curve of constant slope (hence a helix on the unit sphere). We take $\kappa = \pm\omega \sin \mu s$ and continue to designate the tangent indicatrix by subscript t. Arclength along the curve and along its tangent indicatrix are related by

$$\frac{ds_t}{ds} = \kappa = \pm\omega \sin \mu s,$$

so

$$s_t = \mp\frac{\omega}{\mu} \cos \mu s + C.$$

Taking the lower signs, $C = 0$, and $\alpha = |\mathbf{A}| = \sqrt{\omega^2 + \mu^2}$, while substituting $r = 1$ and $\cos\theta = \mu/\alpha$ into the formula for s_t in Section 3, we obtain

$$s_t = \frac{\omega}{\mu} \cos \frac{\mu}{\alpha - \mu}\psi$$

hence

$$s = \frac{1}{\alpha - \mu}\psi,$$

giving a remarkably simple reparametrization

$$\begin{aligned}
x'(s) &= x_t(s) = \frac{\alpha + \mu}{2\alpha}\cos(\alpha - \mu)s - \frac{\alpha - \mu}{2\alpha}\cos(\alpha + \mu)s\\
y'(s) &= y_t(s) = \frac{\alpha + \mu}{2\alpha}\sin(\alpha - \mu)s - \frac{\alpha - \mu}{2\alpha}\sin(\alpha + \mu)s\\
z'(s) &= z_t(s) = \frac{\omega}{\alpha}\cos \mu s.
\end{aligned}$$

Theorem 2. *An arclength parametrization of a curve of constant precession with natural equations* $\kappa(s) = -\omega \sin \mu s$ *and* $\tau(s) = \omega \cos \mu s$ *is given by*

$$x(s) = \frac{\alpha+\mu}{2\alpha}\frac{\sin(\alpha-\mu)s}{\alpha-\mu} - \frac{\alpha-\mu}{2\alpha}\frac{\sin(\alpha+\mu)s}{\alpha+\mu}$$

$$y(s) = -\frac{\alpha+\mu}{2\alpha}\frac{\cos(\alpha-\mu)s}{\alpha-\mu} + \frac{\alpha-\mu}{2\alpha}\frac{\cos(\alpha+\mu)s}{\alpha+\mu}$$

$$z(s) = \frac{\omega}{\mu\alpha}\sin \mu s$$

where ω, μ, *and* $\alpha = \sqrt{\omega^2+\mu^2}$ *are constant. Moreover, the curve lies on the circular hyperboloid of one sheet*

$$x^2 + y^2 - \frac{\mu^2}{\omega^2}z^2 = \frac{4\mu^2}{\omega^4}.$$

The curve is closed if and only if μ/α *is rational.*

A curve of constant precession is rendered in Figure 2. The tangent indicatrix, a spherical helix, has cusps where $\kappa(s) = -\omega \sin \mu s = 0$.

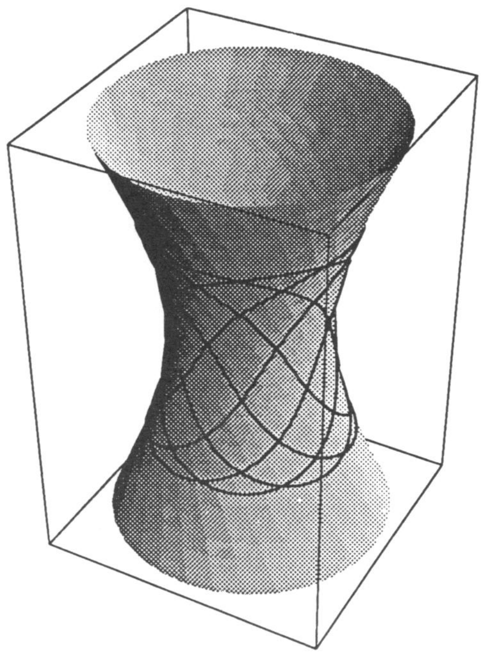

Figure 2. A curve of constant precession with $\omega = 15$ and $\mu = 8$, shown on its circular hyperboloid. It is an integral curve of the indicatrix in Figure 1.

REFERENCES

1. B.-Y. Chen, Some open problems and conjectures on submanifolds of finite type, *Soochow J. Math.* 17 (1991), 169–188.
2. B.-Y. Chen, J. Deprez, F. Dillen, L. Verstraelen, and L. Vrancken, Curves of finite type, in *Geometry and Topology of Submanifolds, II*, World Scientific, Singapore, 1990, pp. 76–110.

3. J. Deprez, F. Dillen, and L. Vrancken, Finite type curves on quadrics, *Chinese J. Math* 18 (1990), 95–121.
4. N. V. Efimov, Nekotorye zadachi iz teorii prostranstvennykh krivykh, *Usp. Mat. Nauk* 2 (1947), 193–194.
5. A. Enneper, Zur Theorie der Curven doppelter Krümmung, *Math. Ann.* 19 (1882), 72–83.
6. L. Euler, De constructione aequationum ope motus tractorii aliisque ad methodum tangentium inversam pertinentibus, *Comm. Acad. Sci. Petrop.* 8 (1736), 66–85. Reprinted in *Leonhardi Euleri Opera Omnia, Ser. I*, Vol. 22, Basel, 1926, pp. 83–107.
7. W. Fenchel, The differential geometry of closed space curves, *Bull. Amer. Math. Soc.* 57 (1951), 44–54.
8. F. Frenet, Sur les courbes à double courbure, *Jour. de Math.* 17 (1852), 437–447. Extrait d'une Thèse (Toulouse, 1847).
9. R. Hoppe, Ueber die Darstellung der Curven durch Krümmung und Torsion, *J. Reine Angew. Math.* 60 (1862), 182–187.
10. C.-C. Hwang, A differential-geometric criterion for a space curve to be closed, *Proc. Amer. Math. Soc.* 83 (1981), 357–361.
11. M. A. Lancret, Mémoire sur les courbes à double courbure, *Mém. des sav. étrangers* 1 (1806), 416–454.
12. K. Nomizu, On Frenet equations for curves of clsss C^∞, *Tôhoku Math. J.* (2) 11 (1959), 106–112.
13. W. Schmeidler, Notwendige und hinreichende Bedingungen dafür, daß eine Raumkurve geschlossen ist, *Arch. Math.* 7 (1956), 384–385.
14. D. J. Struik, *Lectures on Classical Differential Geometry*, Dover, New York, 1988. Reprint of second edition (Reading, 1961).
15. Y.-C. Wong and H.-F. Lai, A critical examination of the theory of curves in three dimensional geometry, *Tôhoku Math. J.* (2) 19 (1967), 1–31.

Department of Mathematics
Washington and Lee University
Lexington, VA 24450
scofield@wlu.edu

Don't talk to me of your Archimedes' lever. He was an absentminded person with a mathematical imagination. Mathematics commands all my respect, but I have no use for engines. Give me the right word and the right accent and I will move the world.

—*Joseph Conrad*

Preface to *A Personal Record*. Garden City NY: Doubleday, Doran and Co. Inc., 1929, p. xiii.

Answer to Picture Puzzle
(p. 530)

Carl Ludwig Siegel and Grahame Segal.